ASYMPTOTIC OF THE FREE VIBRATIONS OF A CLAMPED RECTANGULAR PLATE. FORMULATION OF THE SHORTENED PROBLEM

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The edge effect at corner points is not taken into account when applying the asymptotic method [1-3] to the analysis of natural vibrations of rectangular plates. We show how to refine the accuracy of construction of the asymptotic with the corner boundary layers taking into account [4, 5]. Since the variability of the main part of the solution and the edge effect is of an identical order, it is impossible to write the boundary conditions of the original problem [1-3] in canonical form [6], and consequently, the method of elimination is used [7, 8].

The original problem of free vibrations of a clamped rectangular plate is divided into two lower-order problems. The first describes the oscillating main part of the solution, while the second is considered as a perturbation problem, where a small parameter is associated with a large eigenvalue.

Formulation of the reduced problem permits reduction of the investigation of the asymptotic of the eigenfunctions and eigenvalues of the original problem to a study of the eigenfunctions and eigenvalues of the reduced problem, which is of lower order.

1. Formulation of the Problem. Asymptotic Expansions.

The asymptotic of the natural vibrations modes and frequencies of a constant-thickness clamped plate is considered in a rectangular domain $0 \le x \le a$, $0 \le y \le b$

$$\Delta\Delta w(x, y) - \omega^2 k^2 w(x, y) = 0, \tag{1.1}$$

where w(x, y) is the normal deflection, ω is the natural vibration frequency, $k^2 = \rho h/D$, h is the plate thickness, ρ is the specific mass, and D is the cylindrical stiffness. The boundary conditions are

$$w|_\Gamma = 0, \quad \left.\frac{\partial w}{\partial n}\right|_\Gamma = 0. \tag{1.2}$$

Equation (1.1) has the representation

$$(\Delta + \omega k)(\varepsilon^2\Delta - k)w(x, y) = 0, \ \varepsilon^2 = \omega^{-1}.$$

Let $\omega \gg 1$, then $\varepsilon \ll 1$. Let us assume that w(x, y) = u(x, y) + v(x, y). The functions u(x, y) and v(x, y) satisfy the equations

$$\Delta u(x, y) + \omega k u(x, y) = 0; \tag{1.3}$$

$$\varepsilon^2 \Delta v(x, y) - k v(x, y) = 0. \tag{1.4}$$

Equation (1.3) yields the main oscillating part of the solution of (1.1) and (1.4) the rapidly damped part, i.e., the edge effect during vibrations.

We seek the solution of (1.4) in a form proposed in [4]

$$v(x, y) = \sum_{i=0}^{\infty} \varepsilon^i \, [\Pi_{1i}(\xi_1, y) + \Pi_{2i}(\xi_2, y) + Q_{1i}(x, \eta_1) + Q_{2i}(x, \eta_2) + \\ + P_{1i}(\xi_1, \eta_1) + P_{2i}(\xi_1, \eta_2) + P_{3i}(\xi, \eta_1) + P_{4i}(\xi_2, \eta_2)] \\ (\xi_1 = x/\varepsilon, \ \xi_2 = (a - x)/\varepsilon, \ \eta_1 = y/\varepsilon, \ \eta_2 = (b = y)/\varepsilon). \tag{1.5}$$

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The boundary layer part of the asymptotic consists of boundary functions of two kinds. The ordinary boundary layers Π_{1i}, Π_{2i}, Q_{1i}, Q_{2i}, described by ordinary differential equations and in the neighborhood of each side of the rectangle. For instance, in the neighborhood of the sides $x = 0$, $0 \le y \le b$ the boundary functions $\Pi_{1i}(\xi_1, y)$ are determined sequentially by using the equations

$$\frac{\partial^2\Pi_{1i}}{\partial\xi_1^2} - k\Pi_{1i} = g_i(\xi_1, y) \quad (i = 0, 1, 2, \ldots),$$

where $g_i(\xi_1, y) = -\partial^2\Pi_{1,i-2}/\partial y^2$; $g_0 \equiv g_1 \equiv 0$.

The boundary layer operators in the neighborhood of the other sides are analogous. Later we shall need a specific kind of zero approximation

$$\Pi_{10}(\xi_1, y) = p_0(0, y)\exp(-\sqrt{k}\xi_1); \tag{1.6}$$

$$Q_{10}(x, \eta_1) = p_0(x, 0)\exp(-\sqrt{k}\eta_1). \tag{1.7}$$

Here $p_0(0, y)$, $p_0(x, 0)$ are unknown functions characterizing the boundary layer amplitudes which will be determined in terms of the solution of the reduced problem. The functions Π and Q introduce an additional residual in the boundary conditions in the neighborhood of the corner points by eliminating the residual in the boundary conditions for $u(x, y)$. To eliminate these residuals, a boundary function P is introduced which is determined from elliptic equations. Thus, in the neighborhood of the vertex (0, 0) the boundary functions $P_{1i}(\xi_1, \eta_1)$ are determined sequentially by using the equations

$$(\Delta - k)P_{1i}(\xi_1, \eta_1) = 0, \quad \Delta = \frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\eta_1^2} \quad (i = 0, 1 \ldots) \tag{1.8}$$

in the domain $\xi_1 \ge 0$, $\eta_1 \ge 0$. From the reasoning presented above we have boundary conditions and a condition at infinity for P_{1i}:

$$P_{1i}(\xi_1, \eta_1) \to 0 \text{ when } \xi_1 \to \infty,\ P_{1i}(\xi_1, \eta_1) \to 0 \text{ when } \eta_1 \to \infty,$$

on the boundary $x = 0$

$$P_{1i}(0, \eta_1) = -Q_{1i}(0, \eta_1); \tag{1.9}$$

on the boundary $y = 0$

$$P_{1i}(\xi_1, 0) = -\Pi_{1i}(\xi_1, 0). \tag{1.10}$$

Taking (1.6) and (1.7) into account we obtain boundary conditions for the zero approximation

$$P_{10}(0, \eta_1) = -p_0(0, 0)\exp(-\sqrt{k}\eta_1),$$
$$P_{10}(\xi_1, 0) = -p_0(0, 0)\exp(-\sqrt{k}\xi_1).$$

Exactly as in [4], we change the function in (1.8) that reduces the original problem to a problem with homogeneous boundary conditions

$$P_{10}(\xi_1, \eta_1) = -p_0(0, 0)\exp(-\sqrt{k}(\xi_1 + \eta_1)) + \Omega_0(\xi_1, \eta_1),$$

where the function $\Omega_0(\xi_1, \eta_1)$ is determined from the boundary value problem

$$\begin{gathered}(\Delta - k)\Omega_0(\xi_1, \eta_1) = kp_0(0, 0)\exp(-\sqrt{k}(\xi_1 + \eta_1)),\\ \Omega_0(0, \eta_1) = 0,\ \Omega_0(\xi_1, 0) = 0,\ \Omega_0(\xi_1, \eta_1) \to 0 \text{ for } \xi_1 \to \infty \text{ and } \eta_1 \to \infty.\end{gathered} \tag{1.11}$$

The solution (1.11) using Green's function is presented in [4]

$$\begin{gathered}\Omega_0(\xi, \eta) = \int_0^\infty\int_0^\infty G(\xi, \eta, s, t)\, kp_0(0, 0)\exp(-\sqrt{k}(t + s))\,dt\,ds,\\ G(\xi, \eta, s, t) = \frac{1}{2\pi}[K_0(\sqrt{k}R_1) + K_0(\sqrt{k}R_2) - K_0(\sqrt{k}R_3) - K_0(\sqrt{k}R_4)].\end{gathered} \tag{1.12}$$

Here $K_0(z)$ is a cylindrical function of imaginary argument

$$R_1 = \sqrt{(\xi - t)^2 + (\eta - s)^2};\ R_2 = \sqrt{(\xi + t)^2 + (\eta + s)^2};$$
$$R_3 = \sqrt{(\xi - t)^2 + (\eta + s)^2};\ R_4 = \sqrt{(\xi + t)^2 + (\eta - s)^2}.$$

Estimates for the function $\Omega_0(\xi_1, \eta_1)$ and its derivatives are established in [4, 5]

$$\left|\Omega_0,\ \frac{\partial\Omega_0}{\partial\xi_1},\ \frac{\partial\Omega_0}{\partial\eta_1}\right| \leqslant C\exp(-\varkappa\rho_0), \tag{1.13}$$

where $\rho_0 = \sqrt{\xi_1^2 + \eta_1^2}$; $0 < \kappa \leq \sqrt{k}$; C and κ are arbitrary constants. For the subsequent terms P_{1i} (i = 1, 2, ...) the boundary conditions have the form

$$P_{1i}(0, \eta_1) = \beta_i(\eta_1)\exp(-\sqrt{k}\eta_1),\ P_{1i}(\xi_1, 0) = \alpha_i(\xi_1)\exp(-\sqrt{k}\xi_1)$$

[$\alpha_i(\xi_1)$ and $\beta_i(\eta_1)$ are polynomials in ξ_1 and η_1]. And since the boundary conditions are consistent at the corner point, then $\alpha_i(0) = \beta_i(0)$. Executing the appropriate substitution in (1.8)

$$P_{1i}(\xi_1, \eta_1) = \Omega_i(\xi_1, \eta_1) + [\alpha_i(\xi_1) + \beta_i(\eta_1) - \alpha_i(0)]\exp(-\sqrt{k}(\xi_1 + \eta_1)),$$

we obtain a boundary value problem analogous to (1.11) for $\Omega_i(\xi_1, \eta_1)$. The functions Ω_1 and their derivatives have exponential estimates of the form (1.13). The functions P_{2i}, P_{3i}, P_{4i} that play the part of corner boundary layers near the corner points (0, b), (a, 0), (a, b) are constructed analogously.

Therefore, the complete construction of the edge effect (the V. V. Bolotin dynamic edge effect) has been performed by taking account of the corner boundary layers.

2. The Reduced Problem.

We formulate the boundary conditions for (1.3) with respect to the function u(x, y). To do this, we substitute the function w(x, y) = u(x, y) + v(x, y) in the boundary conditions (1.2) [v(x, y) is determined from the expansion (1.5)]. Setting $\varepsilon \ll 1$ in (1.5), we limit ourselves to the zero terms of the expansion. From (1.2)

$$u|_\Gamma = -v|_\Gamma,\quad \left.\frac{\partial u}{\partial n}\right|_\Gamma = -\left.\frac{\partial v}{\partial n}\right|_\Gamma. \tag{2.1}$$

As an example, we consider the boundary x = 0, $0 \leq y \leq b$

$$u(0, y) = -v(0, y),\ u_x(0, y) = -v_x(0, y).$$

Taking (1.6), (1.7) and (1.12) into account, we substitute the representation of the solution (1.5) into the boundary conditions (2.1) and neglecting the mutual influence of the orfinary and corner boundary layers, we obtain on the boundary x = 0

$$u(0, y) = -[\Pi_{10}(0, y) + Q_{10}(0, \eta_1) + Q_{20}(0, \eta_2) + P_{10}(0, \eta_1) + P_{20}(0, \eta_2)] = -p_0(0, y); \tag{2.2}$$

$$\begin{aligned} u_x(0, y) &= -[\Pi_{10,x}(0, y) + Q_{10,x}(0, \eta_1) + Q_{20,x}(0, \eta_2) + P_{10,x}(0, \eta_1) + P_{20,x}(0, \eta_2)] = \frac{\sqrt{k}}{\varepsilon}p_0(0, y) - \frac{\sqrt{k}}{\varepsilon}p_0(0, 0)\left[\exp(-\sqrt{k}\eta_1) + \varepsilon\sqrt{k}\frac{\partial I(0, \eta_1)}{\partial x}\right] - \\ &\quad - \frac{\sqrt{k}}{\varepsilon}p_0(0, b)\left[\exp(-\sqrt{k}\eta_2) + \varepsilon\sqrt{k}\frac{\partial I(0, \eta_2)}{\partial x}\right] - p_{0,x}(0, 0)\exp(-\sqrt{k}\eta_1) - p_{0,x}(0, b)\exp(-\sqrt{k}\eta_2), \end{aligned} \tag{2.3}$$

where $I(\xi, \eta) = \int_0^\infty\int_0^\infty G(\xi, \eta, s, t)\exp[-\sqrt{k}(t + s)]\,dtds$ and $G(\xi, \eta, s, t)$ is determined from (1.12) and the estimates (1.13) are valid.

Eliminating the function $p_0(0, y)$ in (2.2) and (2.3), we find

$$\begin{aligned} \varepsilon u_x(0, y) + \sqrt{k}u(0, y) &= -\sqrt{k}p_0(0, 0)\left[\exp(-\sqrt{k}\eta_1) + \varepsilon\sqrt{k}\frac{\partial I(0, \eta_1)}{\partial x}\right] - \\ &\quad - \sqrt{k}p_0(0, b)\left[\exp(-\sqrt{k}\eta_2) + \varepsilon\sqrt{k}\frac{\partial I(0, \eta_2)}{\partial x}\right] - \varepsilon p_{0,x}(0, 0)\times \\ &\quad \times\exp(-\sqrt{k}\eta_1) - \varepsilon p_{0,x}(0, b)\exp(-\sqrt{k}\eta_2). \end{aligned} \tag{2.4}$$

The quantities $p_0(0, 0)$, $p_{0,x}(0, 0)$ can be determined from the condition of consistency of the boundary conditions at the corner point $u(0, 0) = -v(0, 0)$, $u_x(0, 0) = -v_x(0, 0)$. We use the expansion (1.5) while neglecting the mutual influence of the ordinary and corner boundary layers, when at the corner point (0, 0)

$$u(0, 0) = -[\Pi_{10}(0, 0) + Q_{10}(0, 0) + P_{10}(0, 0)] = -p_0(0, 0). \tag{2.5}$$

We find the remaining functions characterizing the boundary layer amplitudes analogously

$$u(0, b) = -p_0(0, b),\ u_x(0, 0) = -p_{0,x}(0, 0),\ u_x(0, b) = -p_{0,x}(0, b). \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we obtain the boundary condition of (1.3) on the boundary $x = 0$, $0 \leq y \leq b$. Finally, the boundary conditions for the problem (1.3) can be written in the form

$$[\varepsilon u_x(x, y) + \sqrt{k}u(x, y)]_{x=x_j} = \sum_{i=1}^{2}\left\{\sqrt{k}u(x_j, y_i)\left[\exp(-\sqrt{k}\eta_i) + \right.\right. \tag{2.7}$$

$$\left.\left. + \sqrt{k}\frac{\partial I(0, \eta_i)}{\partial \xi_j}\right] + \varepsilon u_x(x_j, y_i)\exp(-\sqrt{k}\eta_i)\right\} \quad (j = 1, 2);$$

$$[\varepsilon u_y(x, y) + \sqrt{k}u(x, y)]_{y=y_j} = \sum_{i=1}^{2}\left\{\sqrt{k}u(x_i, y_j)\left[\exp(-\sqrt{k}\xi_i) + \right.\right. \tag{2.8}$$

$$\left.\left. + \sqrt{k}\frac{\partial I(\xi_i, 0)}{\partial \eta_j}\right] + \varepsilon u_y(x_i, y_j)\exp(-\sqrt{k}\xi_i)\right\} \quad (j = 1, 2),$$

$$x_1 = 0,\quad x_2 = a,\quad y_1 = 0,\quad y_2 = b.$$

We therefore obtain a generalized eigenfunction and eigennumber problem (1.3), (2.7), (2.8), which we call the reduced problem of the original problem (1.1) and (1.2).

Assume the solutions of (1.3), (2.7), (2.8) are given. The edge effects in the representation (1.5) can now be restored. Thus, from (2.5) and (2.6) we find the value of the function $p_0(x, y)$ and its derivatives at the corner points (0, 0), (0, b), (a, 0), (a, b) in terms of the function $u(x, y)$. Substituting the value of $p_0(x, y)$ at the corner points into the solutions describing the boundary layer, we determine them completely. Furthermore, substituting the functions $u(x, y)$ and $P_i(\xi, \eta)$ into (2.2) and analogous expressions for the other boundaries, we find the functions $p_0(0, y)$, $p_0(a, y)$, $p_0(x, 0)$, $p_0(x, b)$ and the edge effects on the boundaries in terms of them. Thus, the construction of the functions $u(x, y)$ and $v(x, y)$ is completed fully, and therefore, the solution of the original problem is obtained.

We go over to a discussion of the possible simplifications by following [1-3]. Using (1.13), the right sides in (2.7) and (2.8) have the estimates

$$[\varepsilon u_x(x, y) + \sqrt{k}u(x, y)]_{x=x_j} \leqslant C_{1j}\exp(-\varkappa\eta_1) + C_{2j}\exp(-\varkappa\eta_2),$$
$$[\varepsilon u_y(x, y) + \sqrt{k}u(x, y)]_{y=y_j} \leqslant D_{1j}\exp(-\varkappa\xi_1) + D_{2j}\exp(-\varkappa\xi_2), \tag{2.9}$$

where $0 < \kappa \leq \sqrt{\kappa}$, C_{ij}, D_{ij} are constants. From (2.9) it results that the right sides of (2.7) and (2.8) differ substantially from zero only near the corner points.

If the secondary parts in the equalities (2.7) and (2.8) are neglected, then we obtain the solution of the original problem (1.1) and (1.2) that agrees with the solution of the problem proposed in [1-3]. The influence of the corner boundary layers is not taken into account in the method of [1-3].

The reduced problem is formulated in the case of the existence of axes of symmetry relative to the boundary conditions. The main part of the solution is symmetric relative to these same axes of symmetry, and after simplification, the separation of the spatial variables is possible; this is in good agreement with the results in [9]. In the absence of axes of symmetry relative to the boundary conditions, the formulation of the reduced problem is complicated substantially since the boundary conditions at the corner points can be discontinuous and the construction of the corner boundary layers is made substantially more difficult.

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COROTATION DERIVATIVES AND DEFINING RELATIONS IN THE THEORY OF LARGE PLASTIC STRAINS

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Interest in elastoplasticity with large strains, by which here we shall mean deformations with strain gradients exceeding (componentwise) 0.1, has increased markedly in the last decade. The main problem addressed by the theory of large elastoplastic strains is the derivation of the defining relations, in whose formulation some types of objective differential measures of the stressed and strained states, called corotational in the literature published abroad, are widely used. In this paper corotational derivatives are defined in a unified manner, and A. A. Il'yushin's theory of elastoplastic processes is extended to the case of large plastic deformations.

1. In what follows we shall require indifferent tensors. For this, following [1], we introduce two motions $r(\xi^i, t)$ and $r'(\xi^i, t)$ of the volume of the continuous medium under study differing by a rigid displacement:

$$\mathbf{r}'(\xi^i, t) = \mathbf{p}'(t) + [\mathbf{r}(\xi^i, t) - \mathbf{p}(t)]\cdot\mathbf{O}(t). \tag{1.1}$$

Here $p(t)$ is the radius vector of a particle, chosen as the pole, in the motion $r(\xi^i, t)$, $p'(t)$ is the pole in the motion $r'(\xi^i, t)$; $O(t)$ is a properly orthogonal tensor; (ξ^i, t) are Lagrangian variables. We shall denote the reference configuration by $\mathscr{K}_0$, and the actual configuration in the motions r and r' by $\mathscr{K}_t$ and $\mathscr{K}_t'$, respectively. The basis vectors in $\mathscr{K}_0$ are $\overset{\circ}{e}_i = \partial R_0/\partial\xi^i$ and the basis vectors in $\mathscr{K}_t$ and $\mathscr{K}_t'$ are

$$\widehat{\mathbf{e}}_i = \frac{\partial \mathbf{r}}{\partial \xi^i}, \quad \widehat{\mathbf{e}}_i' = \frac{\partial \mathbf{r}'}{\partial \xi^i}, \quad i = \overline{1, 3}.$$

From (1.1)

$$\widehat{\mathbf{e}}_i' = \widehat{\mathbf{e}}_i\cdot\mathbf{O} = \mathbf{O}^T\cdot\widehat{\mathbf{e}}_i, \quad \widehat{\mathbf{e}}_i = \widehat{\mathbf{e}}_i'\cdot\mathbf{O}^T = \mathbf{O}\cdot\widehat{\mathbf{e}}_i'.$$

Analogous relations also follow from the properties of the orthogonal tensor for the vectors of the conjugate basis:

$$\widehat{\mathbf{e}}'^i = \widehat{\mathbf{e}}^i\cdot\mathbf{O} = \mathbf{O}^T\cdot\widehat{\mathbf{e}}^i, \quad \widehat{\mathbf{e}}^i = \widehat{\mathbf{e}}'^i\cdot\mathbf{O}^T = \mathbf{O}\cdot\widehat{\mathbf{e}}'^i.$$

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